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Nonexpansive selections of metric projections in spaces of continuous functions

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Abstract

A subset A of a metric space X is said to be a nonexpansive proximinal retract (NPR) of X if the metric projection from X to A admits a nonexpansive selection. We study the structure of NPR's in the space $C(K)$ of continuous functions on a compact Hausdorff space K. The main results are a characterization of finite-codimensional and of finite-dimensional NPR subspaces of $C(K)$ and a complete characterization of all NPR *subsets* of l_∞^n.

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1. Introduction

A subset A of a metric space X is said to be proximinal if the metric projection of every point $x \in X$ (i.e., the set $P_A(x)$ of points in A nearest to x) is nonempty. Proximinal sets, their structure and the existence of single-valued selections for the multi-valued metric projection have been the subject of a lot of research. Note that a continuous single-valued selection for the metric projection is a retraction of X onto A. Another family of retracts, the nonexpansive retracts (i.e., subsets $A \subset X$ such that there is a nonexpansive retraction from X onto A), has also been the subject of intensive study.

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In this article, we combine these two properties and study sets $A \subset X$ for which there is a map from X onto A which is simultaneously a single-valued selection of the metric projection and a nonexpansive map. We call such sets nonexpansive proximinal retracts and the associated map will be called a nonexpansive proximinal retraction. (We shall abbreviate both as NPR.)

As the title suggests, our main interest in this article is when the containing space X is a $C(K)$ space.

We use standard notation. In particular, we shall identify $C(K)^*$ with the space of regular Borel measures on K. We only consider real Banach spaces, although many of the results extend to the complex case.

We shall use without further explanation some basic properties of nonexpansive (not necessarily proximinal) retracts $A \subset X$. It is clear that such a set A is closed. If X is a convex subset of a normed space, then A is metrically convex. Indeed, if $\varphi : X \to A$ is a nonexpansive retraction and if x, y are two points in A, then the curve $\gamma(t) = \varphi((1-t)x + ty)$ (for $0 \leqslant t \leqslant 1$) connects x and y in A. By the nonexpansiveness of φ and the triangle inequality this curve is a "metric segment": $\|\gamma(t) - \gamma(s)\| = |t-s| \, \|x-y\|$.

In Section 2 we consider NPR subspaces of $C(K)$ spaces. We characterize their finite-codimensional and finite-dimensional NPR subspaces and formulate a conjecture on the characterization of a general NPR subspace of $C(K)$. The results are analogous to the results on linear selections for the metric projection, see for example [3,5], although the methods and proofs are, of course, different.

In Section 3 we consider the case of finite-dimensional $C(K)$ spaces, namely, the spaces l_∞^n. For these spaces we give a complete characterization of NPR *subsets* (and not only subspaces as in Section 2): they are exactly the intersections of NPR half-spaces. In particular, it turns out that NPR subsets of l_∞^n are convex. We do not know if this is true in general $C(K)$ spaces, but we give an example showing that in general Banach spaces a NPR subset does not have to be convex.

In this section, we use the fact that l_∞^n is a hyperconvex space and apply the following theorem from [7]. For the sake of the reader, and since the article [7] uses a somewhat different terminology, we give the proof of the theorem, as well as basic information on hyperconvex spaces, in the Appendix.

Theorem 1.1 (*Espínola et al. [7]*). *A boundedly compact subset A of a hyperconvex metric space X is a NPR of X if and only if A is a NPR of* $A \cup \{z\}$ *for any* $z \in X \setminus A$.

We finish the introduction with the comment that in many cases the existence of a nonexpansive retraction from a Banach space X onto a closed subspace E implies the existence of a norm-one linear projection on E. This is the case, for example when E is reflexive, or is norm-one complemented in its second dual (see [4, Chapter 7]).

A simpler observation of this nature (explicitly stated in Aronszajn and Smith [2], but possibly even older), is that when E is a proximinal one-codimensional subspace of E, then the metric projection admits a linear selection.

The existence of a linear norm-one projection gives some information on the geometry of E that could be used to study its structure (although we shall not use such an approach in this article). But it should be noted that when E is a NPR, then even if a norm-one linear projection P does exist, P is usually not proximinal. (A linear projection P is a NPR iff it is bi-contractive, i.e., $\|P\| = \|I - P\| = 1$.) Indeed, the one-dimensional subspace of $C(K)$ consisting of the constant functions is a NPR (take $S = K$ for a subspace of type II, see Section 2). Also by the Hahn–Banach theorem every one-dimensional subspace of a Banach space is the range of a norm-one projection. But one checks easily that when K has at least three points, then this subspace is not the range of a linear bi-contractive projection.

2. NPR subspaces of spaces of continuous functions

We start by describing three types of canonical NPR subspaces of $C(K)$:
Type I: Fix a clopen (closed and open) subset $Z \subset K$ and put

$$E_Z^0 = \{f \in C(K) : f_{|Z} \equiv 0\}.$$

A nonexpansive proximinal retraction onto E_Z^0 is given by

$$\varphi(f)(t) = \begin{cases} 0 & \text{for } t \in Z, \\ f(t) & \text{for } t \notin Z. \end{cases}$$

Type II: Fix a clopen subset $S \subset K$ and put

$$E_S = \{f \in C(K) : f_{|S} \text{ is constant}\}.$$

A nonexpansive proximinal retraction onto H_S is given by

$$\varphi(f)(t) = \begin{cases} (\max_{s\in S} f(s) + \min_{s\in S} f(s))/2 & \text{for } t \in S, \\ f(t) & \text{for } t \notin S. \end{cases}$$

Type III: Fix two disjoint clopen subsets $S^1, S^2 \subset K$ and put

$$E_{S^1,S^2} = \{f \in C(K) : f_{|S^i} \text{ is constant and } f_{|S^1} = -f_{|S^2}\}.$$

E_{S^1,S^2} is a NPR because the isometry T of $C(K)$ onto itself given by

$$Tf = \begin{cases} f & \text{on } S^1, \\ -f & \text{on } K \setminus S^1, \end{cases}$$

maps E_{S^1,S^2} onto the NPR subspace E_S, where $S = S^1 \cup S^2$.

It is obvious that translates of these subspaces are also NPR's. Also, when these subspaces are of codimension one (i.e., when the sets Z, S^1, S^2 reduce to single points and S to two points), then these retractions are actually linear. (This is true for E_Z^0 without the restriction that it is one-codimensional.)

It should also be noted that a subspace of codimension one is a NPR iff the half-spaces it determines are NPR.

Using these canonical NPR subspaces, we now describe more NPR subspaces. Let Z, $\{S_i\}_{i=1}^n$ and $\{S_j^1, S_j^2\}_{j=1}^m$ be a *finite* family of mutually disjoint clopen sets and put

$$\begin{aligned} E &= \{f \in C(K) : f_{|Z} = 0, \text{ and } f_{|S_i}, f_{|S_j^1} = -f_{|S_j^2} \text{ are constant}\} \\ &= E_Z^0 \cap \left(\cap E_{S_i}\right)\left(\cap E_{S_j^1,S_j^2}\right). \end{aligned} \tag{1}$$

Then one checks easily that E is also a NPR (with the natural formula for the retraction).

Note that E is finite-dimensional iff the union of the disjoint sets Z, S_i, S_j^1, S_j^2 has a finite complement in K.

The main results of this section are the following two theorems.

Theorem 2.1. *Let E be a finite-codimensional NPR subspace of* $C(K)$, *then it has the form* (1).

Theorem 2.2. *Let E be a finite-dimensional NPR subspace of* $C(K)$, *then it has the form* (1).

We do not know whether the dimension restrictions in these theorems are really necessary. We conjecture they are not:

Conjecture 2.3. *Every NPR subspace of a* $C(K)$ *space is of the form* (1).

Theorems 2.1 and 2.2 show, in particular, that when K is connected, then $C(K)$ has no finite-codimensional or finite-dimensional NPR subspaces except for the one-dimensional subspace consisting of the constant functions (i.e., E_K). If Conjecture 2.3 is true, then this is actually the only NPR subspace it has.

Before passing to the proof of Theorem 2.1, we first need some preparations.

Lemma 2.4. *Let E be a NPR subspace of* $C(K)$ *of finite codimension. Then*

(i) *Every measure in the annihilator* $E^\perp$ *is purely atomic.*
(ii) *If* $k \in K$ *is an atom of some measure* $\mu \in E^\perp$, *then k is isolated in K.*

Proof. Let $\varphi : C(K) \to E$ be the NPR.

Let $\eta_1, \dots, \eta_n$ be a basis for $E^\perp$ and put $\eta = |\eta_1| + \cdots + |\eta_n|$. Denote the (countable) set of atoms of η by $\mathcal{A} \subset K$ and note that $\mathcal{A}$ contains all the atoms of any $\mu \in B(E^\perp)$. Also, for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ so that $\eta(A) < \delta$ implies that $|\mu|(A) < \varepsilon$ for every $\mu \in B(E^\perp)$.

We are now ready for the proofs.

(i) Fix $\tau \in E^\perp$ and $\varepsilon > 0$. By the regularity of τ there are two disjoint compact sets K^+ and K^- contained in the supports of the positive and negative parts $\tau^\pm$ of τ, respectively, with $|\tau|(K^+ \cup K^-) > \|\tau\| - \varepsilon$. Let v be the restriction of τ to $K^+ \cup K^-$ and let f be a continuous function with $-1 \leqslant f \leqslant 1$ such that $f_{K^+} \equiv 1$ and $f_{K^-} \equiv -1$. Thus $|f| \equiv 1$ a.e.-dv and $f\,dv$ is a nonnegative measure. Clearly $\|v - \tau\| < \varepsilon$.

Note that $\|\varphi(f) - f\| = d(f, E) \leqslant \|f\| = 1$, and thus $\varphi(f)(k) \geqslant 0$ on K^+, where $f(k) = 1$. Similarly $\varphi(f)(k) \leqslant 0$ on K^-. It follows that $\varphi(f)\,dv$ is also a nonnegative measure. Also $\|\varphi(f)\| = \|\varphi(f) - \varphi(0)\| \leqslant \|f\| = 1$.

Fix now any point $k \in K^+ \setminus \mathcal{A}$. Since k is not an atom of η we can find, by the equi-integrability of $B(E^\perp)$, an open neighborhood V of k with $\overline{V} \subset \{f > 1 - \varepsilon\}$ so that $|\mu|(V) < \varepsilon$ for every $\mu \in B(E^\perp)$. Let $0 \leqslant g \leqslant 1$ be a continuous function supported in V so that $g(k) = 1$. It follows that

$$\|f - 2g\| \leqslant 1 + \varepsilon. \tag{2}$$

We claim that $\|2g - \varphi(2g)\| \leqslant 2\varepsilon$. Indeed, by the choice of V we obtain that $|\int g\,d\mu| < \varepsilon$ for every $\mu \in B(E^\perp)$. Identifying $\big(C(K)/E\big)^*$ with $E^\perp$ and using the definition of the norm in $C(K)/E$, it follows that there is a $h \in E$ with $\|g - h\| \leqslant \varepsilon$. Thus $\|2g - \varphi(2g)\| = d(2g, E) \leqslant \|2g - 2h\| \leqslant 2\varepsilon$.

Combining this estimate with (2), it follows that

$$\begin{aligned} |\varphi(f)(k) - 2| = |\varphi(f)(k) - 2g(k)| &\leqslant \|\varphi(f) - \varphi(2g)\| + \|\varphi(2g) - 2g\| \\ &\leqslant \|f - 2g\| + 2\varepsilon \leqslant 1 + 3\varepsilon. \end{aligned}$$

Thus $\varphi(f)(k) \geqslant 1 - 3\varepsilon$. Using also $f(k) = 1$ and $\|\varphi(f)\| \leqslant 1$ give that $|f(k) - \varphi(f)(k)| \leqslant 3\varepsilon$. Similarly, $|f(k) - \varphi(f)(k)| \leqslant 3\varepsilon$ when $k \in K^- \setminus \mathcal{A}$. Since v is supported in $K^+ \cup K^-$ and $f\, dv$ is nonnegative, it follows that

$$\int_{K\setminus\mathcal{A}} \varphi(f)\, dv \geqslant \int_{K\setminus\mathcal{A}} f\, dv - 3\varepsilon = |v|(K \setminus \mathcal{A}) - 3\varepsilon.$$

Using $\tau \in E^{\perp}$, $\|v - \tau\| < \varepsilon$ and $\int_{\mathcal{A}} \varphi(f)\, dv \geqslant 0$ (because, $\varphi(f)\, dv$ is a nonnegative measure), it follows that

$$0 = \int \varphi(f)\, d\tau \geqslant \int \varphi(f)\, dv - \varepsilon = \int_{\mathcal{A}} \varphi(f)\, dv + \int_{K\setminus\mathcal{A}} \varphi(f)\, dv - \varepsilon$$
$$\geqslant |v|(K \setminus \mathcal{A}) - 4\varepsilon.$$

Thus $|\tau|(K \setminus \mathcal{A}) \leqslant |v|(K \setminus \mathcal{A}) + \varepsilon \leqslant 5\varepsilon$. Letting $\varepsilon \to 0$ gives that $|\tau|(K \setminus \mathcal{A}) = 0$, i.e., that τ is purely atomic.

(ii) Denote the atoms of η by $\mathcal{A} = \{k_j\}$. As observed earlier, the atoms of any $\mu \in B(E^{\perp})$ are contained in $\mathcal{A}$.

Assume $v \in B(E^{\perp})$ has an atom at a nonisolated point, say, at k_1. Normalize so that $\|v\| = 1$, put $v(k_j) = v^j$, and assume that $v^1 > 0$. Fix $\varepsilon > 0$.

Choose N so that $\sum_{j>N} |\mu(k_j)| < \varepsilon$ for every $\mu \in B(E^{\perp})$ and let V be a neighborhood k_1, such that $k_j \notin \overline{V}$ for $2 \leqslant j \leqslant N$. Let $-1 \leqslant f \leqslant 1$ be a continuous function with $f \equiv 1$ in V and $f(k_j) = \operatorname{sign}(v^j)$ for $2 \leqslant j \leqslant N$. As in part (i) we obtain that $\|\varphi(f)\| \leqslant 1$ and that $\varphi(f)(k_j)v^j \geqslant 0$ for every $j \leqslant N$.

Since k_1 is not isolated, every neighborhood $U \subset V$ of k_1 contains a point $k_U \neq k_1, \ldots, k_N$. Choose a continuous $0 \leqslant g \leqslant 1$ supported in U with $g(k_U) = 1$ and $g(k_1) = 0$. Thus $g(k_j) = 0$ for $j \leqslant N$ and $\|f - 2g\| = 1$. Since

$$\left|\int g\, d\mu\right| = \left|\sum_{j>N} g(k_j)\mu(k_j)\right| \leqslant \sum_{j>N} |\mu(k_j)| < \varepsilon$$

for every $\mu \in B(E^{\perp})$, it follows, as in part (i), that $|\varphi(f)(k_U) - 2| \leqslant 1 + 3\varepsilon$ and consequently that $\varphi(f)(k_U) \geqslant 1 - 3\varepsilon$. But the neighborhood U was arbitrary, hence also $\varphi(f)(k_1) \geqslant 1 - 3\varepsilon$. Thus

$$0 = \int \varphi(f)\, dv = \varphi(f)(k_1)v^1 + \sum_{2 \leqslant j \leqslant N} \varphi(f)(k_j)v^j + \sum_{j>N} \varphi(f)(k_j)v^j.$$

But $\varphi(f)(k_1)v^1 \geqslant (1 - 3\varepsilon)v^1 > 0$, the first sum is nonnegative and the second is bounded in absolute value by ε. This is impossible when ε is so small that $(1 - 3\varepsilon)v^1 > \varepsilon$. $\square$

Proof of Theorem 2.1. We first observe that it is enough to prove the theorem under the additional assumption that E is not contained in any "canonical" hyperplane or, equivalently

(*) *$E^{\perp}$ does not contain any measure of the form δ_k or $\delta_k \pm \delta_l$.*

Of course, under (*) we need to show that actually $E = C(K)$.

The reduction to this special case is obtained as follows: assume that there is a point $z \in K$ with $f(z) = 0$ for all $f \in E$. By the lemma z is isolated in K, hence $F = \{f \in C(K) : f(z) = 0\}$ is isometric to $C(K \setminus \{z\})$ and $E \subset F$. The restriction of φ to F is a NPR from F onto E. Similarly, if there are isolated points $k \neq l$ in K so that $f(k) = f(l)$ (resp., $f(k) = -f(l)$) for all $f \in E$, then E is contained in $F = \{f \in C(K) : f(k) = f(l)\}$ (resp., $f(k) = -f(l)$), which is isometric to $C(K \setminus \{l\})$, and again the restriction of φ to F is a NPR from F onto E.

Making these reductions at most n times (where n is the codimension of E), yields the required reduction.

Before passing to the proof, we make the useful observation that when we are given a measure $\mu = \sum \mu^j \delta_{k_j} \in E^\perp$ and a finite set J of indices, then we may assume that $\mu^j \geqslant 0$ for all $j \in J$. Indeed, assume that $\mu^j < 0$ for some $j \in J$. Since k_j is isolated, the operator T that changes the sign of a function f at the point k_j is an isometry of $C(K)$ onto itself with $T^{-1} = T$. We can thus replace E by TE, the retraction φ by $T \circ \varphi \circ T$, and the atom μ^j of μ at k_j by $-\mu^j$.

Assume now for contradiction that E satisfies (*) and that its codimension is $n \geqslant 1$. By Lemma 2.4 every $\mu \in E^\perp$ is purely atomic and there is a countable set of isolated points $\{k_j\}$ containing all the atoms of elements in $E^\perp$.

Find a basis $\mu_1, \ldots, \mu_n$ for $E^\perp$ which, after possibly renumbering of the k_j's, has the form

$$\mu_i = \delta_{k_i} + \sum_{j>n} \mu_i^j \delta_{k_j} \quad \text{for } i \leqslant n.$$

Fix $\varepsilon > 0$ and choose $N > n$ so that $\sum_{j>N} |\mu_i^j| < \varepsilon$ for all $1 \leqslant i \leqslant n$. The function f on K defined by $f(k_j) = 1$ for $1 \leqslant j \leqslant N$ and $f(k) = 0$ otherwise is continuous because the k_j's are isolated. As in Lemma 2.4, $\varphi(f)(k_j) \geqslant 0$ for all $1 \leqslant j \leqslant N$ and $\|\varphi(f)\| \leqslant 1$.

Claim. $\sum_{j>n} |\mu_i^j| \leqslant 1$ *for all* $1 \leqslant i \leqslant n$.

Assume that $\max_{i \leqslant n} \sum_{n<j \leqslant N} |\mu_i^j|$ is attained for $i = 1$, and we show that it is bounded by 1. Since this holds for every large enough N the claim will follow.

Put $\lambda_i = \sum_{n<j\leqslant N} \mu_i^j$. As noted above, we may assume that $\mu_1^j \geqslant 0$ for every $n < j \leqslant N$, hence $\lambda_1 = \sum_{n<j\leqslant N} \mu_1^j \geqslant 0$. We may also assume that $\lambda_i \geqslant 0$ for every $i \geqslant 2$ (by replacing, if necessary, μ_i by $-\mu_i$ and changing the sign of $\mu_i(k_i)$).

With this notation we need to prove that $\lambda_1 \leqslant 1$, so assume for contradiction that $\lambda_1 > 1$ and define g by

$$g(k) = \begin{cases} -\lambda_i & \text{for } k = k_i \text{ and } i \leqslant n, \\ 1 & \text{for } k = k_j \text{ and } n < j \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

Once again g is continuous because the k_i's are isolated. Also $g \in E$ because the definition of g and the λ_i's imply that $\int g \, d\mu_i = 0$ for all $i \leqslant n$.

The nonzero values of the function $f - tg$ are $1 + t\lambda_i$ for $i \leqslant n$ and $1 - t$. It follows from $\lambda_1 > 1$, the maximality of λ_1 and from $\lambda_i \geqslant 0$ for all $i \leqslant n$ that if $t < 0$ and if $|t|$ is large enough, then

$$\|f - tg\| = \max\{|1 - t|, |1 + \lambda_i t|\} = |1 + \lambda_1 t| = -1 - \lambda_1 t.$$

Combining this estimate with $\varphi(tg) = tg$ and $t < 0$ it follows that if $|t|$ is large enough, then

$$\begin{aligned} -\big(\varphi(f)(k_1) + \lambda_1 t\big) &= |\varphi(f)(k_1) + \lambda_1 t| = \big|\big(\varphi(f) - \varphi(tg)\big)(k_1)\big| \\ &\leqslant \|f - tg\| = -1 - \lambda_1 t \end{aligned}$$

and hence $\varphi(f)(k_1) \geqslant 1$. But this is impossible for small enough ε, because

$$0 = \int \varphi(f)\, d\mu_1 = \varphi(f)(k_1) + \sum_{n<j\leqslant N} \mu_1^j \varphi(f)(k_j) + \sum_{j>N} \mu_1^j \varphi(f)(k_j)$$

and $\varphi(f)(k_1) \geqslant 1$, the first sum is nonnegative (because $\varphi(f)(k_j)$ and μ_1^j are nonnegative for all $n < j \leqslant N$) and the third term is bounded in absolute value by ε (because $\sum_{j>N} |\mu_1^j| < \varepsilon$ and $\|\varphi(f)\| \leqslant 1$). This proves the claim.

Combining the claim with the assumption (*), it follows that $|\mu_i^j| < 1$ for all $i \leqslant n$ and $j > n$, and that for each $i \leqslant n$ there is a $j > n$ with $\mu_i^j \neq 0$. Assume that $0 < \mu_1^{n+1} < 1$, say, and then assume also that $\mu_1^j \geqslant 0$ for $n+2 \leqslant j \leqslant N$. We may also assume that $\mu_i^{n+1} \geqslant 0$ for every $i \geqslant 2$ (by replacing, if necessary, the measure μ_i by $-\mu_i$ and changing the sign of $\mu_i(k_i)$).

Let $f \in C(K)$ be as above (i.e. $f(k_j) = 1$ for $j \leqslant N$ and $f(k) = 0$ otherwise), then $\|\varphi(f)\| \leqslant 1$ and $\varphi(f)(k_j) \geqslant 0$ for $j \leqslant N$. Define $g \in E$ by $g(k_i) = -\mu_i^{n+1}$ for $i \leqslant n$, $g(k_{n+1}) = 1$ and $g(k) = 0$ otherwise.

The nonzero values of $f - tg$ are $1 + t\mu_i^{n+1}$ at k_i for $i \leqslant n$, $1 - t$ at k_{n+1} and the value 1. Since $0 \leqslant \mu_i^{n+1} < 1$ for every $i \leqslant n$, it follows that if $t > 0$ is large enough, then $\|f - tg\| = |1 - t| = t - 1$. Thus, if $t > 0$ is large enough, then

$$\begin{aligned} 0 &\leqslant t - \varphi(f)(k_{n+1}) = \big(\varphi(tg) - \varphi(f)\big)(k_{n+1}) \\ &\leqslant \|\varphi(tg) - \varphi(f)\| \leqslant \|tg - f\| = t - 1 \end{aligned}$$

and hence $\varphi(f)(k_{n+1}) \geqslant 1$. But this is impossible for small enough ε because

$$0 = \int \varphi(f)\, d\mu_1 = \sum_{1\leqslant j\leqslant N;\, j\neq n+1} \mu_1^j \varphi(f)(k_j) + \mu_1^{n+1} \varphi(f)(k_{n+1}) + \sum_{j>N} \mu_1^j \varphi(f)(k_j),$$

where the first term is nonnegative, the second at least $\mu_1^{n+1} > 0$, and the third is bounded in absolute value by ε. □

Corollary 2.5. *If K is perfect (i.e., with no isolated points), then $C(K)$ does not admit any NPR subspace of finite codimension.*

For the proof of Theorem 2.2 we shall need the following known lemma.

Lemma 2.6. *Let E be a subspace of $C(K)$ which is the range of a nonexpansive retraction $\psi : C(K) \to E$. Then E^* is isometric to $L_1(\mu)$ for some measure μ.*

Proof. Lindenstrauss [8, Theorem 6.1, (2) $\Leftrightarrow$ (12)] proved that E^* is isometric to $L_1(\mu)$ iff every collection of four mutually intersecting balls in E with the same radius r has a common intersection.

If $B_E(x_i, r)$ are the four balls in E, then the balls $B_{C(K)}(x_i, r)$ in $C(K)$ with the same centers and radius intersect in $C(K)$, because $C(K)^*$ is isometric to an $L_1(\mu)$ space. Choose a point f in their intersection, then $\psi(f) \in \cap B_E(x_i, r)$. □

Proof of Theorem 2.2. Since E is the range of a nonexpansive retraction of $C(K)$, it follows from Lemma 2.6 that E^* is isometric to a finite-dimensional $L_1(\mu)$ space, i.e., to l_1^n. Thus E is isometric to l_∞^n.

Let $\{f_i\}_{i=1}^n \subset E$ be a l_∞^n basis for E, i.e., $\|\sum_{i\leqslant n}\alpha_i f_i\| = \max_{i\leqslant n}|\alpha_i|$ for all scalars $\{\alpha_i\}_{i\leqslant n}$. It follows that the sets $S_i = \{t \in K : |f_i(t)| = 1\}$ are nonempty and pairwise disjoint. (Actually S_i is disjoint from $\{t : f_j(t) \neq 0\}$ whenever $i \neq j$). Also $\sum_{i\leqslant n}|f_i(t)|\leqslant 1$ for all $t \in K$. Put $S = \cup_{i\leqslant n}S_i$.

The theorem will follow once we show that $f_i(t) = 0$ for all i and for all $t \notin S$. Indeed, take $Z = K \setminus S$, the sets S_i for the i's where f_i has a constant sign on S_i, and $S_i^1 = \{t \in S_i : f_i(t) = 1\}$ and $S_i^2 = \{t \in S_i : f_i(t) = -1\}$ for the i's where f_i attains both values ± 1 on S_i. The continuity of the f_i's implies that all these sets are clopen.

Thus, assume for contradiction that there is a $t_1 \notin S$ so that $f_1(t_1) \neq 0$, say.

Put $I = \{i : f_i(t_1) \neq 0\}$. Replacing f_i by $-f_i$ if necessary, we may assume that $f_i(t_1) > 0$ for all $i \in I$.

Pick $1 > \eta > f_1(t_1)$ and set $T = \{|f_1(t)|\geqslant\eta\}$. Then T contains S_1 and is disjoint from $(\cup_{i\neq 1}S_i) \cup \{t_1\}$. Using Tietze's theorem, find $f \in C(K)$ with $\|f\| = 1$ so that

$$f(t) = \begin{cases} f_1(t) & \text{for } t \in T, \\ f_i(t) & \text{for } t \in S_i;\ 1 \neq i \in I, \\ -1 & \text{for } t = t_1 \end{cases}$$

and expand $\varphi(f) = \sum \alpha_i f_i$. We claim that $\alpha_1 = 0$.

Indeed, fix $i \in I$ and $t \in S_i$. Then $\|\varphi(f) - f\| = d(f, E)\leqslant\|f\| = 1$ and $f(t) = f_i(t) = \pm 1$, together with $f_j(t) = 0$ for $j \neq i$ imply that $\alpha_i \geqslant 0$. Since $f_i(t_1) > 0$ for all $i \in I$ by our normalization and since $\alpha_i \geqslant 0$, we obtain that $\varphi(f)(t_1) = \sum_{i\in I}\alpha_i f_i(t_1)\geqslant 0$ and is strictly positive if one of the α_i's is nonzero. But then $f(t_1) = -1$ and $|\varphi(f)(t_1) - f(t_1)|\leqslant 1$ implies that necessarily $\varphi(f)(t_1)\leqslant 0$, hence $\varphi(f)(t_1) = 0$ and $\alpha_i = 0$ for all $i \in I$. In particular $\alpha_1 = 0$ as claimed.

Fix $\lambda > 1$ and $s \in S_1$. Then $\alpha_1 = 0$ and $f_i(s) = 0$ for all $i \neq 1$ imply that

$$\|\varphi(f) - \varphi(\lambda f_1)\| = \left\|\sum_{i\neq 1}\alpha_i f_i - \lambda f_1\right\| \geqslant \left|\sum_{i\neq 1}\alpha_i f_i(s) - \lambda f_1(s)\right| = |0 - \lambda f_1(s)| = \lambda.$$

We finish the proof by showing that $\|f - \lambda f_1\| < \lambda$ for big enough λ, contradicting the nonexpansiveness of φ. To this end we distinguish two cases:

If $t \in T$, then $f(t) = f_1(t)$, hence

$$|(f - \lambda f_1)(t)| = |(1-\lambda)f_1(t)|\leqslant\lambda - 1 < \lambda.$$

If $t \notin T$, then $|f_1(t)|\leqslant\eta$, hence

$$|(f - \lambda f_1)(t)|\leqslant|f(t)| + \lambda|f_1(t)|\leqslant 1 + \lambda\eta < \lambda$$

provided $\lambda > 1/(1-\eta)$. □

Corollary 2.7. *If K is connected, then $C(K)$ does not admit any NPR subspace of finite dimension except for the one-dimensional subspace E_K of type II.*

3. NPR subsets of l_∞^n

The main result of this section is a complete characterization of NPR subsets of l_∞^n.

Theorem 3.1. *A subset $A \subset l_\infty^n$ is a NPR iff it is the intersection of NPR half-spaces.*

We also give some results in general Banach spaces and make some comments on the structure of NPR's in general $C(K)$ spaces. We start with some preliminary preparations.

Lemma 3.2. *Let A be a convex NPR in a Banach space X and assume that the affine subspace E spanned by A is finite-dimensional. Then*

(i) *E is a NPR of X.*
(ii) *Let z be a smooth point of the relative boundary of A in E and let V be the supporting hyperplane of A in E at the point z. Then V^+, the half-space of E determined by V and containing A, is a NPR of X.*

Proof. Let $\varphi : X \to A$ be the NPR and assume, as we may, that $0 \in A$. Direct computation shows that for each $\lambda > 0$ the map $\varphi_\lambda(x) = \lambda\varphi(x/\lambda)$ is a NPR from X onto λA, and for each fixed x the function $\lambda \to \varphi_\lambda(x)$ is bounded by $\|x\|$ (because $\varphi_\lambda(0) = 0$).

Since E is finite-dimensional, there is a E-valued Banach limit LIM on bounded function from $\mathbb{R}^+$ to E. One checks easily that $\psi(x) = \mathrm{LIM}_{\lambda\to\infty}\varphi_\lambda(x)$ is a NPR from X onto the closure Y of $\cup\{\lambda A : \lambda > 0\}$.

To prove (i) assume that 0 is in the relative interior of A in E. It then follows that $Y = E$.

To prove (ii) assume that the smooth point is $z = 0$. It follows from the smoothness that $Y = V^+$. □

Remark. The assumption that E is finite-dimensional could, of course, be replaced by weaker conditions. What we really need is that closed balls in E are compact under some topology $\mathcal{T}$ so that the norm is lower semi-continuous with respect to $\mathcal{T}$. (For example, the ω^*-topology when E happens to be a dual space.) We shall use, however, only the finite-dimensional case.

Lemma 3.3. *Let $A \subseteq l_\infty^n$ be a NPR in l_∞^n. Then A is convex.*

Proof. Denote the NPR on A by $\varphi : l_\infty^n \to A$.

Observe first that whenever a point $v = (v_1, \ldots, v_n) \in l_\infty$ attains its norm in all its coordinates, i.e., when $|v_i|$ is constant, then the linear segment connecting v and $-v$ is the only metric segment between them.

We shall show that whenever there is a point $x \in A$ so that also $-x \in A$, then $0 \in A$. The general case follows by translation.

Choose x so that it attains its norm in k coordinates, and so that k is maximal among all the points $y \in A$ with $-y \in A$. We shall show that $k = n$, and this will prove the lemma: Since A is a NPR, any two points in A are connected in A by a metric segment, and by the observation above $k = n$ implies that the metric segment connecting x and $-x$ is a linear segment. Hence $0 \in A$.

Assume for contradiction that $k < n$. We may assume that $\|x\| = 1$ and that $x = (a_1, \ldots, a_n)$ with $a_j \geqslant 0$ and $a_1 = \cdots = a_k = 1$. Put $\max\{a_j : j > k\} = \alpha < 1$ and $x_t = (t, \ldots, t, a_{k+1}, \ldots, a_n)$ for $\alpha \leqslant t \leqslant 1$. Note that x_α attains its norm ($\|x_\alpha\| = \alpha$) in at least $k + 1$ coordinates. We claim that $x_\alpha \in A$, and a similar argument will show that $-x_\alpha \in A$. This contradicts the maximality of k.

Assume the claim is false. Since A is closed there is an $\varepsilon > 0$ so that $B(x_\alpha, \varepsilon) \cap A = \emptyset$. Let $[\alpha, s)$ be the maximal interval so that $B(x_t, \varepsilon) \cap A = \emptyset$ for all $\alpha \leqslant t < s$.

Since $\|x_t - x\| = 1-t$ and $x \in A$, it follows that if $B(x_t, \varepsilon) \cap A = \emptyset$, then $\varepsilon < 1-t$. Taking the supremum over $\alpha \leqslant t < s$ gives that $s \leqslant 1-\varepsilon$. Also $d(x_s, A) = \varepsilon$ implies that $\varphi(x_s) \in B(x_s, \varepsilon) \cap A$.

Observe also that if $y \in B(x_s, \varepsilon) \cap A$, then there is an $i \leqslant k$ so that $y_i = s + \varepsilon$. Indeed, $s - \varepsilon \leqslant y_j \leqslant s + \varepsilon$ for all $j \leqslant k$. Since $y \notin B(x_t, \varepsilon)$ for $\alpha \leqslant t < s$, then the two conditions $y \in B(x_s, \varepsilon)$ and $(x_s)_j = (x_t)_j$ for $j > k$ imply that there is an $i \leqslant k$ so that either $y_i > t + \varepsilon$ or $y_i < t - \varepsilon$. But the latter is impossible because combining $y_i < t - \varepsilon$ with $y_i \geqslant s - \varepsilon$ would contradict $t < s$. Letting $t \to s$ gives $y_i \geqslant s + \varepsilon$ and proves the observation.

Applying the observation above to $y = \varphi(x_s) \in B(x_s, \varepsilon) \cap A$, choose $i \leqslant k$ so that $\big(\varphi(x_s)\big)_i = s + \varepsilon$. Then

$$\|\varphi(x_s) - \varphi(-x)\| = \|\varphi(x_s) - (-x)\| \geqslant \Big(\varphi(x_s) - (-x)\Big)_i = s + \varepsilon + 1$$

but on the other hand

$$\|x_s - (-x)\| = \max\Big(s + 1, 2\max\{a_j : j > k\}\Big) = s + 1$$

because $2\max\{a_j : j > k\} = 2\alpha \leqslant 2s < 1 + s$. This contradicts the nonexpansiveness of φ and proves the lemma. □

We do not know if NPR's in infinite-dimensional $C(K)$ spaces are necessarily convex. The following example shows, however, that NPR's do not have to be convex in general Banach spaces.

Example 3.4. Let E be the two-dimensional Banach space whose unit ball is the regular hexagon with vertices at $(\pm 2/\sqrt{3}, 0)$; $(\pm 1/\sqrt{3}, \pm 1)$. Let $A \subset E$ be the (nonconvex) union of the two rays emanating from the origin and passing through $(1/\sqrt{3}, \pm 1)$. One checks directly that if $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then $\|(x_1, x_2) - (y_1, y_2)\| \geqslant |x_2 - y_2|$ and that $\|(x_1, x_2) - (y_1, y_2)\| = |x_2 - y_2|$ whenever $x, y \in A$. It follows that the horizontal projection $\varphi(x) = (|x_2|/\sqrt{3}, x_2)$ from E onto A is nonexpansive, and one checks directly that $\varphi(x)$ is a nearest point in A to x.

Proof of Theorem 3.1. Assume that A is a NPR in l_∞^n and we show that it is the intersection of NPR half-spaces.

Let E be the affine subspace of l_∞^n spanned by A and we may assume that $0 \in A$, i.e., that E is a linear subspace. By Lemma 3.3 the set A is convex, hence part (i) of Lemma 3.2 applies and E is a NPR. By Theorem 2.2 E is isometric l_∞^k for some $k \leqslant n$. Moreover, the explicit form (1) of NPR subspaces implies that E is the intersection of NPR hyperplanes in l_∞^n.

Since the smooth points of the relative boundary of A in E are dense in this boundary, it follows that every $y \in E \setminus A$ can be separated from A by a hyperplane in E which supports A in a relatively smooth point. By part (ii) of Lemma 3.2 the half-space determined by this hyperplane is a NPR in E, and the special form (1) of E implies that it is the intersection of E with a NPR half-space of l_∞^n. Thus A is, indeed, the intersection of NPR half-spaces.

Conversely, assume that A is an intersection of NPR half-spaces in l_∞^n. The special form (1) of the NPR hyperplanes in l_∞^n is applied through the following claim:

Claim. *Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two points in A and let $t \geqslant 0$. Define a new point $c = (c_1, c_2, \dots, c_n) = c(a, b, t)$ by*

$$c_i = \begin{cases} a_i & \text{if } |a_i - b_i| \leqslant t, \\ b_i - t & \text{if } a_i < b_i - t, \\ b_i + t & \text{if } a_i > b_i + t. \end{cases}$$

Then also $c \in A$.

To prove the claim we may assume that A is just one NPR half-space and we check separately each of the three types of NPR half-spaces. Thus assume, for example, that $A = \{x = (x_1 \dots, x_n) : x_1 + x_2 \leqslant 1\}$ is a half-space of type II and we make a case by case check that $c_1 + c_2 \leqslant \max(a_1 + a_2, b_1 + b_2) \leqslant 1$. For example, assume that $c_1 = a_1$ and $c_2 = b_2 + t$. Then $b_2 + t < a_2$, and hence $c_1 + c_2 < a_1 + a_2$. The other cases, as well as checking the other types of half-spaces are similar.

We shall apply the claim for a pair of points satisfying $\|b\| \geqslant \|a\| = 1$ and with $t = \|b\| - 1$. Then certainly $\|c\| \geqslant \|b\| - t = 1$ and actually $\|c\| = 1$. Indeed, assume for example that $a_1 < b_1 - t$. Then $c_1 = b_1 - t = b_1 - \|b\| + 1 \leqslant 1$, and clearly $c_1 = b_1 - t > a_1 \geqslant -\|a\| = -1$, hence $|c_1| \leqslant 1$. Similar estimates show that $|c_i| \leqslant 1$ for all i and in the other cases as well, hence $\|c\| \leqslant 1$.

Moreover, with $\|a\|$, $\|b\|$ and t as above the estimate $\|b - c\| \leqslant t = \|b\| - \|c\|$ together with the triangle inequality give that $\|b - c\| = \|b\| - \|c\|$. Hence

$$\|c\| - \|a - c\| \leqslant \|c\| - (\|a - b\| - \|b - c\|) = \|b\| - \|b - a\|. \tag{3}$$

We now prove by induction on the dimension n that A is a NPR.

Since l_∞^n is hyperconvex, Theorem 1.1 (which is proved in the Appendix) implies that it suffices to prove that for every $z \in l_\infty^n \setminus A$ the set A is a NPR in $A \cup \{z\}$. We thus need to find a point $a \in A$, which is nearest to z in A, and such that $\|a - b\| \leqslant \|z - b\|$ for every $b \in A$.

We may assume that $z = 0$ and that $\mathrm{dist}(0, A) = 1$. Since A is convex, its intersection with the unit ball B of l_∞^n is contained in a face of B. We may assume that the face is $B \cap H$, where $H = \{x : x_1 = 1\}$. In particular $\|b\| \geqslant 1$ for all $b \in A$. Let $R : l_\infty^n \to H$ be the nonexpansive retraction $R(x_1, x_2, \dots) = (1, x_2, \dots)$ and note that $R(0) = e_1$.

Since H is a translate of l_∞^{n-1}, the induction hypothesis implies that there is a NPR $\varphi : H \to H \cap A$. Put $a = \varphi(e_1) = (\varphi R)(0)$ and note that $a \in H \cap B$. Hence, $\|a\| = 1$ and it is a nearest point in A to $z = 0$.

To show that $\|a - b\| \leqslant \|0 - b\| = \|b\|$ for every $b \in A$ (or, equivalently, that $\|b\| - \|b - a\| \geqslant 0$), let $c = c(a, b, t)$ with $t = \|b\| - 1$ be as above. Then $\|c\| = 1$ and $c \in A$ imply that it is in the face of B determined by H, i.e., $c \in B \cap A \subset H \cap A$ and hence $(\varphi R)(c) = \varphi(c) = c$.

Then $\|c - a\| = \|(\varphi R)(c) - (\varphi R)(0)\| \leqslant \|c - 0\| = \|c\|$, because φR is nonexpansive. Combined with (3) this gives $\|b\| - \|b - a\| \geqslant \|c\| - \|a - c\| \geqslant 0$ as required. □

Remarks. (i) Lemmas 3.2 and 3.3 hold also when A is a NPR of a neighborhood B of A (rather than the whole space l_∞^n). It follows that if $A \subset l_\infty^n$ is a NPR of such a neighborhood B, then A is the intersection of NPR half-spaces and, in particular a NPR of all of l_∞^n.

(ii) Lemmas 3.2 and 3.3 also remain true when A is a NPR of a NPR subset $B \subset l_\infty^n$. We leave it to the reader to check that this, indeed, follows from the special form of such a set B as an intersection of NPR half-spaces of l_∞^n. Thus a NPR subset $A \subset B$ of a NPR set $B \subset l_\infty^n$ is a NPR in l_∞^n. This is no longer true in general Banach spaces.

Example 3.5. Let A be the nonconvex NPR subset in the two-dimensional space E of Example 3.4. Denote the hexagon by H.

Let D be the unit disk in $\mathbb{R}^2$. Then D is the inscribed disk in H. Let F be the three-dimensional space whose unit ball B is the convex hull of H and $\{(x_1, x_2, \pm 1) : (x_1, x_2) \in D\}$. Denote by $P : F \to E$ the projection given by $P(x_1, x_2, x_3) = (x_1, x_2, 0)$. One checks easily that $\|P\| = \|I - P\| = 1$, and thus P is a NPR from F onto E.

We shall show that A is not a NPR of F. In fact the metric projection from F to A does not admit any continuous selection.

Consider the points $x_t = (\frac{2}{\sqrt{3}} + |t|, \sqrt{3}t, 1)$, and let $B_t = B(x_t, 1)$ be the closed ball in F of radius 1 and center x_t. Then the intersection $B_t \cap E$ is the translated disk $(\frac{2}{\sqrt{3}} + |t|, \sqrt{3}t) + D$, which touches A in a unique point whenever $t \neq 0$. This point is the nearest point in A to x_t. As $t \to 0^+$ and $t \to 0^-$ we get two different limit points: the two points where the disk $(\frac{2}{\sqrt{3}}, 0) + D$ touches A. (These two points are exactly the nearest points in A to x_0.) Thus the metric projection from F to A does not admit a selection which is continuous at x_0.

Remarks. We make a few comments on the analogs of Theorem 3.1 in general $C(K)$ spaces.

(i) If A is a *finite* intersection of NPR half-spaces in any $C(K)$ space, then it is a NPR. Indeed the explicit form (1) of NPR hyperplanes implies that there is a finite clopen subset $S \subset K$ of cardinality n, say, and a subset $B \subset C(S) = l_\infty^n$, which is an intersection of NPR hyperplanes in $C(S)$, so that $A = \{f \in C(K) : f_{|S} \in B\}$. By Theorem 3.1 there is a NPR $\psi : C(S) \to B$, and then the map $\varphi : C(K) \to A$, given by $\varphi(f)(k) = \psi(f_{|S})(k)$ when $k \in S$ and $\varphi(f)(k) = f(k)$ otherwise, is a NPR on A.

(ii) An infinite intersection of NPR hyperplanes does not have to be a NPR. For example, assume that K contains a convergent sequence $\{k_n\}$ of isolated points with limit k, and take $E_n = \{f \in C(K) : f(k_{2n}) = 0\}$. Then $E = \cap E_n$ does not admit any nonexpansive retraction φ (not even necessarily NPR). Indeed, let e be the constant function 1. Since $f(k) = 0$ for every $f \in E$, it follows that $\varphi(e)(k) = 0$ and we can find n such that $|\varphi(e)(k_{2n+1})| < \frac{1}{2}$. Let g be the (continuous) function taking the value 1 at k_{2n+1} and 0 elsewhere. Then $g \in E$ and $\|e - 2g\| = 1$, yet

$$\|\varphi(e) - \varphi(2g)\| = \|\varphi(e) - 2g\| \geqslant |\varphi(e)(k_{2n+1}) - 2g(k_{2n+1})| > 3/2.$$

(iii) It is also false that a NPR subset of an infinite-dimensional $C(K)$ needs to be an intersection of NPR half-spaces. Indeed, for any K the set $C(K)^+ = \{f \in C(K) : f \geqslant 0\}$ is a NPR with associated retraction $\varphi(f) = \max\{f, 0\}$. But when K is connected $C(K)$ admits no NPR hyperplane whatsoever.

Similarly, Ubhaya [9] proved (among other results) that the set of nondecreasing continuous functions on $C(0, 1)$ is a NPR. He also showed that for each fixed $M > 0$ and $0 < \alpha \leqslant 1$, the set of all $f \in C(0, 1)$, such that $|f(x) - f(y)| \leqslant M|x - y|^\alpha$ is a NPR. Once again, $C(0, 1)$ admits no NPR hyperplane because the interval is connected.

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Appendix A. Proof of Theorem 1.1

Recall that a metric space X is called hyperconvex if every family $\{B(x_i, r_i)\}_{i\in I}$ of balls in X satisfying $d(x_i, x_j) \leqslant r_i + r_j$ has a common intersection. An equivalent condition is that when Y is any metric space containing X, then there is a nonexpansive retraction from Y onto X. The systematic study of hyperconvex spaces and the relations between intersection properties of balls and extensions of maps was initiated by Aronszajn and Panitchpakdi [1]. See Espínola and Khamsi [6] for details on hyperconvex spaces.

Hyperconvex Banach spaces are exactly the $C(K)$ spaces with K an extremally disconnected compact Hausdorff space. In particular, the finite-dimensional hyperconvex Banach spaces are exactly the spaces l_∞^n.

We now turn to the proof of Theorem 1.1, which is Theorem 4.1 in [7]. (The formulation in [7] is for compact sets A, but the proof holds for boundedly compact sets.) Lemma A.1 below and the proof of Theorem 1.1 combine the proofs of Theorem 2.1, Lemma 2.2 and Theorem 4.1 in [7]. (Subsets A satisfying the conclusion of the following lemma were called in [7] weakly externally hyperconvex.)

Lemma A.1. *Let A be a subset of a hyperconvex metric space X so that for every $y \in X$ there is a NPR from $A \cup \{y\}$ onto A. Then for every family of mutually intersecting balls $\{B_i\}_{i\in I}$ with centers in A and for every point $z \in X \setminus A$ so that $B_i \cap B(z, d(z, A)) \neq \emptyset$ for every i, the intersection $\left(\bigcap_i B_i\right) \cap B(z, d(z, A)) \cap A$ is nonempty.*

Proof. Put $B_i = B(x_i, r_i)$, where $x_i \in A$, and set $r_z = d(z, A)$.

By hyperconvexity the intersection $B = \left(\bigcap_i B_i\right) \cap B(z, r_z)$ is nonempty, and we need to show it intersects A. Since $B \subset B(z, r_z)$, we actually need to show that it intersects $A_1 = A \cap B(z, r_z)$. Choose $a \in A_1$ and $b \in B$ with $d(a, b) < \frac{3}{2} d(A_1, B)$ and put $d(a, b) = 2d$. We shall prove that $d = 0$.

One checks easily that the balls $B(a, d)$, $B(b, d)$ and $B(z, r_z - d)$ are mutually intersecting. By the hyperconvexity of X there is a point y with

$$y \in B(a, d) \cap B(b, d) \cap B(z, r_z - d).$$

Let $\varphi : A \cup \{y\} \to A$ be a NPR and note first that $\varphi(y) \in A_1$. Indeed, we only need to check that $\varphi(y) \in B(z, r_z)$, but

$$d(\varphi(y), z) \leqslant d(\varphi(y), y) + d(y, z) \leqslant d(y, A) + r_z - d \leqslant r_z$$

because $d(y, A) \leqslant d(y, a) \leqslant d$.

Next we show that there is a point $x \in B$ with $d(\varphi(y), x) \leqslant d$. Indeed, $d(\varphi(y), z) \leqslant r_z$, the estimate

$$d(\varphi(y), x_i) = d(\varphi(y), \varphi(x_i)) \leqslant d(y, x_i) \leqslant d(y, b) + d(b, x_i) \leqslant d + r_i$$

and the fact that $B_i \cap B(z, r_z) \neq \emptyset$ for all i imply, by the hyperconvexity of X, that the balls $B(\varphi(y), d)$, $B(x_i, r_i)$ and $B(z, r_z)$ have a common intersection, i.e., that there is a point $x \in B(z, r_z) \cap \left(\bigcap_i B(x_i, r_i)\right) = B$ with $d(x, \varphi(y)) \leqslant d$.

It follows that $2d = d(a,b) < \frac{3}{2}d(A_1, B) \leqslant \frac{3}{2}d(\varphi(y), x) \leqslant \frac{3d}{2}$ and hence $d = 0$. $\square$

Proof of Theorem 1.1. We shall show that for every finite set $F \subset X \setminus A$ there is a NPR from $A \cup F$ onto F. The theorem then follows by a standard compactness argument, using the compactness of bounded sets in A.

The proof is by induction on the cardinality of F. Choose $z \in F$ such that $d(z, A) = \max_{y\in F} d(y, A)$ and set $G = F \setminus \{z\}$. Let $\varphi : A \cup G \to A$ be a NPR. The family of balls

$$\{B(x, d(x, z)) : x \in A\}\ ;\ \ \{B(\varphi(y), d(y, z)) : y \in G\}\ ;\ \ B(z, d(z, A))$$

satisfies the conditions of Lemma A.1, where the assumption that $d(z, A)$ is maximal is used to check that $d(\varphi(y), z) \leqslant d(y, z) + d(z, A)$ for $y \in G$. Indeed, $d(\varphi(y), z) \leqslant d(\varphi(y), y) + d(y, z)$ and $d(\varphi(y), y) = d(y, A) \leqslant d(z, A)$. That the other pairs of balls intersect follows immediately from the triangle inequality or the nonexpansiveness of φ.

By the lemma there is a point $a \in A$ in the intersection of all these balls, and we extend φ to a map on $A \cup F$ by defining $\varphi(z) = a$. $\square$

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